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WAVELET ESTIMATE OF ERROR VARIANCE IN A SEMIPARAMETRIC REGRESSION MODEL WITH MARTINGALE DIFFERENCE ERRORS

HONGCHANG HU

School of Mathematics and Statistics
Hubei Normal University
Huangshi 435002
China
e-mail: retutome@163.com

Abstract

In this paper, consider a semiparametric regression model

$$y_i = X_i^T\beta + g(t_i) + \varepsilon_i,\ 1 \le i \le n,$$

where the error $\{\varepsilon_i,\ F_i,\ 1 \le i \le n\}$ is a martingale difference sequence with $\sigma^2 = Var(\varepsilon_i|F_{i-1})$. We obtain the wavelet estimator of error variance σ^2. Under general conditions, we investigate asymptotic normality of $\hat{\sigma}_n^2$, and the asymptotic chi-square distribution of the quadratic forms in $\hat{\beta}_n$.

1. Introduction

Consider a semiparametric regression model

$$y_i = X_i^T\beta + g(t_i) + \varepsilon_i,\ 1 \le i \le n, \tag{1.1}$$

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where y_i is real-valued relating to the observation at t_i, β is a d-dimensional unknown parameter, $X = (X_{ir})_{n \times d}$ is a random carrier matrix. $g(t)(t \in [0, 1])$ denotes the nonparametric signal, $\{t_i\}$ is a deterministic sequence in interval $[0, 1]$, and $\{\varepsilon_i, F_i, i \le n\}$ is a martingale difference sequence.

Following Speckman [20], denote

$$x_{ir} = f_r(t_i) + \eta_{ir}, 1 \le i \le n, 1 \le r \le d, \tag{1.2}$$

where $f_r(\cdot)$ is a function in interval $[0, 1]$, $\{\overline{\eta}_i, i \ge 1\}$ is a stochastic sequence with $\overline{\eta}_i = (\eta_{i1}, \cdots, \eta_{id})$ i.i.d., and

$$E\overline{\eta}_i = 0, \quad Var(\overline{\eta}_i) = V, \tag{1.3}$$

where $V = (V_{ij})(j = 1, 2, \cdots, d)$ is a positive definite matrix with d-order. Moreover, $\{\eta_{ir}\}$ and $\{\varepsilon_i\}$ are independent.

Since the semiparametric regression model contains linear components and a nonparmetric component, it is more flexible than the usual standard linear models and attractive in some applications. The model was first introduced by Engle et al. [8] and has been extensively studied. For when the errors are independent and identically distributed random variables, Chen [5], Speckman [20], Chen and Shiah [6], Donald and Dewey [7], Hamilton and Truong [9], Bianco and Boente [2], Shi and Teng [19], Qian and Cai [16], Qian et al. [17] and Chai and Xu [3] used various estimation methods (the kernel method, spline method, series estimation, local linear estimation, two-stage estimation, robust estimation and wavelet estimation) to obtain some estimators of the unknown quantities and discussed the asymptotic properties of these estimators.

However, the independence assumption for the errors is not always appropriate in applications, especially for sequentially collected economic data, which often exhibit evident dependence in the errors. Recently, the semiparametric regression model with serially correlated errors has attracted increasing attention from statisticians. One case of considerable interest is the model with martingale difference errors. Chen and Cui [4] given an example to show the application of the semiparametric model with martingale difference errors. Under martingale difference errors, Li and Liu [15] showed that the (weighted) least squares estimators of parameters were strongly consistent and asymptotic normality. Yan et a1. [22] studied consistency of near neighbor estimators. Chen and Cui [4] consider the application of the empirical likelihood method to a partially linear model with martingale difference errors, and shown that the empirical log-likelihood ratio at the true parameter converged to the standard Chi-square distribution. Hu and Hu [10] investigated strong consistency in Model (1)-(3) by the wavelet method. Li and Hu [14] consider the asymptotic normality of the wavelet estimators. Hu [11] studied the semiparametric model with martingale difference linear time series errors, and obtained the r-th mean consistency and completely consistency for the estimators.

In this paper, using the wavelet method, the semiparametric regression model is discussed while the error $\{\varepsilon_i\}$ is a martingale difference sequence. The organization of this paper is as follows: The wavelet estimator of error variance is given in Section 2. Under general conditions, the asymptotic distributions of $\hat{\sigma}_n^2$ and the asymptotic Chi-square distribution of the quadratic forms in $\hat{\beta}_n$ are obtained in Section 3. The main proofs are presented in Section 4.

2. Estimation Method

Suppose that there exists a scaling function $\phi(x)$ in the Schwartz space S_l and a multiresolution analysis $\{V_m\}$ in the concomitant Hilbert space $L^2(R)$, with its reproducing kernel $E_m(t, s)$ given by

$$E_m(t, s) = 2^m E_0(2^m t, 2^m s) = 2^m \sum_{k \in z} \phi(2^m t - k)\phi(2^m s - k).$$

Let $A_i = [s_{i-1}, s_i]$ denote intervals that partition $[0, 1]$ with $t_i \in A_i$ and $1 \leq i \leq n$. The estimate method will be introduced as the following.

Firstly, suppose that β is known, we define estimator of $g(\cdot)$ by

$$\hat{g}_0(t) = \hat{g}_0(t, \beta) = \sum_{i=1}^{n}(y_i - X_i^T\beta)\int_{A_i} E_m(t, s)ds \ ;$$

In succession, we define wavelet estimator $\hat{\beta}_n$ by minimizing

$$\sum_{i=1}^{n}(y_i - X_i^T\beta - \hat{g}_0(t_i, \beta))^2;$$

Finally, we define linear wavelet estimator of $g(\cdot)$ by

$$\hat{g}(t) = \hat{g}_0\left(t, \hat{\beta}\right) = \sum_{i=1}^{n}(y_i - X_i^T\hat{\beta}_n)\int_{A_i} E_m(t, s)ds \ ;$$

Let $X = (X_{ir})_{n\times d}$, $Y = (y_1, \cdots, y_n)^T$, $S = \left(S_{ij}\right)_{n\times n}$, $S_{ij} = \int_{A_j} E_m\,(t_i, s)ds$, $\widetilde{X} = (I - S)X$, $\widetilde{Y} = (I - S)Y$.

Then we obtain that

$$\hat{\beta}_n = (\widetilde{X}^T\widetilde{X})^{-1}\widetilde{X}^T\widetilde{Y}, \ \hat{g}_n = S(Y - X\hat{\beta}),$$

where $\hat{g}_n = (\hat{g}(t_1), \cdots, \hat{g}(t_n))^T$ is an estimator of vector $g = (g(t_1), \cdots, g(t_n))^T$. Thus the wavelet estimator of the error variance is defined by

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - X_i^T\hat{\beta}_n - \hat{g}(t_i))^2,$$

where $\hat{\varepsilon}_i = y_i - X_i^T\hat{\beta}_n - \hat{g}(t_i)$.

To obtain our results, the following four conditions are sufficient.

(A_1) $g(\cdot)$, $f_r(\cdot) \in H^{\alpha}$ (Sobolev space, see Chai and Xu [3]), for some $\alpha > 1/2$, $1 \le r \le d$;

(A_2) $g(\cdot)$ and $f_r(\cdot)$ are Lipschitz functions of order $\gamma > 1/4, 1 \leq r \leq d$;

(A_3) $\phi(\cdot)$ belongs to S_l, which is a Schwartz space for $l \geq \alpha$. ϕ is a Lipschitz function of order 1 and have compact support, in addition to $\left|\hat{\phi}(\xi) - 1\right| = O(\xi)$ as $\xi \to 0$, where $\hat{\phi}$ denotes Fourier transform of ϕ;

(A_4) $s_i (i = 1, \cdots, n)$ and m satisfy $\max_{1 \leq i \leq n}(s_i - s_{i-1}) = O(n^{-1})$ and $2^m = O\left(n^{1/3}\right)$, respectively.

3. Statements of the Results

Now we state the following results of this paper.

Theorem 3.1. *Let* $\{\varepsilon_i, F_i, 1 \leq i \leq n\}$ *be a martingale difference sequence with* $\sup_i E|\varepsilon_i|^4 < \infty$ *and* $\sup_i E|\varepsilon_i| \geq \delta > 0$. *If conditions* (A_1) *-*(A_4) *hold, and* $n^{1/4}\tau_m \to 0$ *as* $n \to \infty$, *then*

$$\sqrt{n}\left(\hat{\sigma}_n^2 - \sigma^2\right) \xrightarrow{L} N\left(0, \breve{\sigma}^4\right), \tag{3.1}$$

where $\breve{\sigma}^4 = Var\left(\varepsilon_i^2 | F_{i-1}\right) < \infty$, $i = 1, 2, \cdots, n$.

Theorem 3.2. *Assume that conditions* $(A_1) - (A_4)$ *hold,* $\{\overline{\eta}_i, 1 \leq i \leq n\}$ *is a measurable random sequence on* $\bigcap_{k=1}^{n} F_{k-1}$, *and* $\{\overline{\eta}_i, 1 \leq i \leq n\}$ and $\{\varepsilon_i, 1 \leq i \leq n\}$ *are a.s. bounded. If there exists some* $u \in (1/2 + 1/q, 1)$, $(q > 2)$ *such that* $\sup_i \int_{A_i} |E_m(t, s)| ds = O(n^{-u})$, *then for* $\sup_i E|\varepsilon_i| \geq \delta > 0$ *and* $n^{1/4}\tau_m \to 0$,

$$\left(\hat{\beta}_n - \beta\right)^T \left(\widetilde{X}^T \widetilde{X}\right)\left(\hat{\beta}_n - \beta\right) \Big/ \sigma^2 \xrightarrow{L} \chi^2(d). \tag{3.2}$$

Remark 3.1. Since σ^2 is unknown, we can not apply the theorem. If we substitute $\hat{\sigma}_n^2$ for σ^2 in (3.2), then we obtain the following Theorem 3.3. So the result can be applied in large sample hypothesis testing.

Theorem 3.3. *Under the conditions of the Theorem* 3.2, *we obtain*

$$\left(\hat{\beta}_n - \beta\right)^T \left(\tilde{X}^T \tilde{X}\right)\left(\hat{\beta}_n - \beta\right) \Big/ \hat{\sigma}_n^2 \xrightarrow{L} \chi^2(d). \tag{3.3}$$

Proof. By Theorem 3.1, we easily obtain $\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$. Therefore, Theorem 3.3 follows from Theorem 3.2.

Remark 3.2. When the errors $\{\varepsilon_i, i = 1, \cdots, n\}$ are i.i.d. random variables, by these theorems, we easily obtain some corresponding results which are discussed by several authors, such as Qian and Cai [16], Qian et al. [17] and Chai and Xu [3].

4. Proofs of Theorems

Before the proofs of the theorems, we introduce some preliminary results. For simplicity, C is a arbitrary positive constant which could take difference value at each occurrence.

Lemma 4.1 (Antoniads et al. [1])**.** *If condition* (A_3) *holds*, *then*

(I) $|E_0(t, s)| \le \dfrac{C_k}{(1 + |t - s|)^k}$ *and* $|E_m(t, s)| \le \dfrac{2^m C_k}{(1 + 2^m |t - s|)^k}$ *for* $k \in N$, *where* C_k *is a real constant depending only on* k ;

(II) $\sup_{0 \le s \le 1} |E_m(t, s)| = O(2^m)$;

(III) $\sup_t \int_0^1 |E_m(t, s)| ds \le C$.

Lemma 4.2 (Hu and Hu [10])**.** *If conditions* (A_1) - (A_4) *hold*, *then*

$$\sup_t \left| f_j(t) - \sum_{k=1}^n \left(\int_{A_k} E_m(t, s) ds\right) f_j(t_k) \right| = O\left(n^{-\gamma}\right) + O(\tau_m)$$

$$\sup_t \left| g(t) - \sum_{k=1}^n \left(\int_{A_k} E_m(t, s) ds\right) g(t_k) \right| = O\left(n^{-\gamma}\right) + O(\tau_m),$$

where $$\tau_m = \begin{cases} 2^{-m(\alpha-1/2)} & 1/2 < \alpha < 3/2, \\ \sqrt{m} \cdot 2^{-m} & \alpha = 3/2, \\ 2^{-m} & \alpha > 3/2. \end{cases}$$

Lemma 4.5. *Let* $\{\xi_{nk}, F_k^n, n, k \geq 1\}$ *be a martingale difference sequence with* $E\xi_{nk}^2 < \infty$. *Assume that* $\sum_{h=1}^{n} E(\xi_{nk}^2 I(|\xi_{nk}| > \delta)|F_{k-1}^n) \xrightarrow{p} 0$ $\forall\delta > 0$ *as* $n \to \infty$ *and* $\sum_{h=1}^{n} E(\xi_{nk}^2|F_{k-1}^n) \xrightarrow{p} \sigma_1^2$ *hold. Then* $\sum_{h=1}^{n} \xi_{nk}$ $\xrightarrow{L} N(0, \sigma_1^2)$.

Proof. See Theorem 1.2 of Kundu et al. [13] or Lemma 1.1 of Hu [12]. □

Lemma 4.5. *Suppose that the* k *-vector* X_n *converge to the* k *-vector* X *in probability, and* $k \times k$ *-order random matrix* B_n *converge to the* $k \times k$ *-order random matrix* B *in probability. Then*

$$X_n B_n X_n^T \xrightarrow{p} XBX^T \quad as \quad n \to \infty.$$

Proof. Let $X_n = (x_{1n}, \cdots, x_{kn})$, $B_n = (b_{ijn})_{k\times k}$, $X = (x_1, \cdots, x_k)$, $B = (b_{ij})_{k\times k}$. Then $x_{in} \xrightarrow{p} x_i$, $b_{ijn} \xrightarrow{p} b_{ij}$, $i, j = 1, \cdots, k$. Hence, we have

$$X_n B_n X_n^T = \sum_{i=1}^{k}\sum_{j=1}^{k} b_{ijn} x_{in} x_{jn} \xrightarrow{p} \sum_{i=1}^{k}\sum_{j=1}^{k} b_{ij} x_i x_j = XBX^T. \qquad \square$$

Remark 4.1. Let B_n be a real-valued matrix sequence, the corresponding result was established by Serfling [18].

Lemma 4.6 (Stout [21]). *Let* $\{X_i, i \geq 1\}$ *be a martingale difference sequence with* $EX_i^2 = 1$ *and* $E|X_i| \geq \delta$ *for some* $\delta > 0$ *and all* $i \geq 1$.

Then, given a sequence of numbers $\{a_i,\ i \geq 1\}$, $\sum_{i=1}^{\infty} a_i X_i$ *a.s. converges if and only if* $\sum_{i=1}^{\infty} a_i^2 < \infty$.

Lemma 4.7 (Li and Hu, [14])**.** *Under the conditions of the Theorem* 3.2, *we obtain*

$$\sqrt{n}\left(\hat{\beta}_n - \beta\right) \xrightarrow{L} N\left(0,\ \sigma^2 V^{-1}\right).$$

Proof of Theorem 3.1. Note that

$$Y - X\hat{\beta}_n - \hat{g}_n = \tilde{Y} - \tilde{X}\hat{\beta}_n = \left(I - \tilde{X}\left(\tilde{X}^T\tilde{X}\right)^{-1}\tilde{X}^T\right)\tilde{Y},$$

we obtain

$$\hat{\sigma}_n^2 = \frac{1}{n}\tilde{Y}^T(I - C_n)^T(I - C_n)\tilde{Y} = \frac{1}{n}\tilde{Y}^T(I - C_n)\tilde{Y}, \tag{4.1}$$

where $C_n = \tilde{X}\left(\tilde{X}^T\tilde{X}\right)^{-1}\tilde{X}^T$. Let $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)^T$ and $\tilde{\varepsilon} = (I - S)\varepsilon$. Then

$$\tilde{Y} = \tilde{X}\beta + \tilde{g} + \tilde{\varepsilon} = \tilde{X}\beta + \tilde{g} + \varepsilon - S(Y - X\beta) = \tilde{X}\beta + (g - \hat{g}_0) + \varepsilon. \tag{4.2}$$

By (4.2), (4.1) and note that $\tilde{X}^T C_n = \tilde{X}^T$, $C_n\tilde{X} = \tilde{X}$, we obtain

$$\begin{aligned}\hat{\sigma}_n^2 - \sigma^2 &= 1/n\left(\beta^T\tilde{X}^T + (g - \hat{g}_0)^T + \varepsilon^T\right)(I - C_n)\left(\tilde{X}\beta + (g - \hat{g}_0) + \varepsilon\right) - \sigma^2 \\ &= 1/n\left((g - \hat{g}_0)^T + \varepsilon^T\right)(I - C_n)\left((g - \hat{g}_0) + \varepsilon\right) - \sigma^2 \\ &= \left[1/n\varepsilon^T(I - C_n)\varepsilon - \sigma^2\right] + \left[1/n(g - \hat{g}_0)^T(I - C_n)(g - \hat{g}_0)\right] \\ &\quad + \left[2/n(g - \hat{g}_0)^T(I - C_n)\varepsilon\right] \hat{=} I_1 + I_2 + I_3. \end{aligned} \tag{4.3}$$

Write

$$I_1 = 1/n\varepsilon^T\varepsilon - \sigma^2 - 1/n\varepsilon^T C_n\varepsilon = \frac{1}{n}\sum_{i=1}^{n}\left(\varepsilon_i^2 - E\left(\varepsilon_i^2 | F_{i-1}\right)\right) - \frac{1}{n}\varepsilon^T C_n\varepsilon \hat{=} I_1^{(1)} - I_1^{(2)}. \tag{4.4}$$

It is easy to show that $\xi_{ni} \hat{=} 1/\sqrt{n}(\varepsilon_i^2 - E(\varepsilon_i^2|F_{i-1}))$ is a martingale difference sequence, and

$$\sum_{i=1}^{n} E\left(\xi_{ni}^2|F_{i-1}\right) = \frac{1}{n}\sum_{i=1}^{n} E\left(\left(\varepsilon_i^2 - E\left(\varepsilon_i^2|F_{i-1}\right)\right)^2|F_{i-1}\right) = \frac{1}{n}\sum_{i=1}^{n} Var\left(\varepsilon_i^2|F_{i-1}\right) = \breve{\sigma}^4. \quad (4.5)$$

By $\sup_i E|\varepsilon_i|^4 < \infty$, we obtain

$$\sum_{i=1}^{n} E\left(\xi_{ni}^2 I(|\xi_{ni}| > \delta)|F_{i-1}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\left(\left(\varepsilon_i^2 - E\left(\varepsilon_i^2|F_{i-1}\right)\right)^2 I\left(\left|\varepsilon_i^2 - E\left(\varepsilon_i^2|\ F_{i-1}\right)\right| > \sqrt{n}\delta\right)\middle|\ F_{i-1}\right)$$

$$\xrightarrow{p} 0,\ n \to \infty,\ \forall\delta \in (0, 1]. \quad (4.6)$$

By Lemma 4.4, we obtain

$$\sqrt{n}I_1^{(1)} \xrightarrow{L} N\left(0, \breve{\sigma}^4\right). \quad (4.7)$$

To complete the proof, we need to prove

$$\sqrt{n}I_1^{(2)} = \left(n^{-3/4}\varepsilon^T \tilde{X}\right)\left(n^{-1}\tilde{X}^T\tilde{X}\right)^{-1}\left(n^{-3/4}\tilde{X}^T\varepsilon\right) \xrightarrow{p} 0. \quad (4.8)$$

It is easy to show that (See Hu [11])

$$\left(n^{-1}\tilde{X}^T\tilde{X}\right)^{-1} \xrightarrow{p} V^{-1}. \quad (4.9)$$

Note that

$$n^{-3/4}\varepsilon^T\tilde{X} = n^{-3/4}\varepsilon^T(I - S)X = n^{-3/4}\varepsilon^T X - n^{-3/4}\varepsilon^T SX = T_1 - T_2. \quad (4.10)$$

The j-th element of $n^{-3/4}\varepsilon^T X$ is given by

$$n^{-3/4}\sum_{i=1}^{n}\varepsilon_i x_{ij} = n^{-3/4}\sum_{i=1}^{n}\varepsilon_i f_j(t_i) + n^{-3/4}\sum_{i=1}^{n}\varepsilon_i\eta_{ij} = T_1^{(1)} - T_1^{(2)}. \quad (4.11)$$

Let $a_i = n^{-7/12} f_j(t_i)\sqrt{E\varepsilon_i^2}$. Then we obtain

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} n^{-7/6} E\varepsilon_i^2 f_j^2(t_i) \leq n^{-1/6}. \sup_i E\varepsilon_i^2. \sup_i f_j^2(t_i) \to 0.$$

Thus $\sum_{i=1}^{\infty} a_i^2 < \infty$. Moreover, by Lemma 4.6, we get

$$\sum_{i=1}^{\infty} a_i \frac{\varepsilon_i}{\sqrt{E\varepsilon_i^2}} = \sum_{i=1}^{n} n^{-7/12} f_j(t_i) \sqrt{E\varepsilon_i^2} \cdot \frac{\varepsilon_i}{\sqrt{E\varepsilon_i^2}} < \infty.$$

So we obtain that

$$T_1^{(1)} = n^{-1/6} \sum_{i=1}^{\infty} a_i \varepsilon_i \Big/ \sqrt{E\varepsilon_i^2} \xrightarrow{p} 0. \tag{4.12}$$

Since η_{ij} and ε_i are independent each other, with $E\eta_{ij} = 0$ and $E|\varepsilon_i| < \infty$, we have $ET_1^{(2)} = n^{-3/4} \sum_{i=1}^{n} E\varepsilon_i \cdot E\eta_{ij} = 0$. Thus

$$T_1^{(2)} \xrightarrow{p} 0. \tag{4.13}$$

From (4.11)-(4.13), we have that

$$T_1 \xrightarrow{p} 0. \tag{4.14}$$

The j-th element of $n^{-3/4} \varepsilon^T SX$ is given by

$$n^{-3/4} \sum_{i=1}^{n} \varepsilon_i \sum_{k=1}^{n} S_{ik} x_{kj} = n^{-3/4} \sum_{i=1}^{n} \varepsilon_i \sum_{k=1}^{n} S_{ik} f_j(t_k) +$$

$$n^{-3/4} \sum_{i=1}^{n} \varepsilon_i \sum_{k=1}^{n} S_{ik} \eta_{kj} = T_2^{(1)} - T_2^{(2)}. \tag{4.15}$$

Similar to (4.14), by Lemma 4.1, we get

$$T_2 \xrightarrow{p} 0. \tag{4.16}$$

From (4.10), (4.14) and (4.16), we have that

$$n^{-3/4}\varepsilon^T \widetilde{X} \xrightarrow{\ p\ } 0. \tag{4.17}$$

Using Lemma 4.5, the (4.8) follows from (4.9) and (4.17). By (4.4), (4.7) and (4.8), we obtain

$$\sqrt{n} I_1 \xrightarrow{\ L\ } N\left(0,\ \breve{\sigma}^4\right). \tag{4.18}$$

Since $I - C_n$ is a symmetry matrix with $(I - C_n)^2 = I - C_n$, and

$$\hat{g}_0(t_i) = \sum_{j=1}^{n}\left(g(t_j) + \varepsilon_j\right)\int_{A_j} E_m(t_i,\ s)ds,$$

we have that

$$I_2 \le 1/n(g - \hat{g}_0)^T(g - \hat{g}_0) = 1/n\sum_{i=1}^{n}(g(t_i) - \hat{g}_0(t_i))^2$$

$$\le \frac{2}{n}\sum_{i=1}^{n}\left(g(t_i) - \sum_{j=1}^{n} g(t_j)\int_{A_j} E_m(t_i, s)ds\right)^2 + \frac{2}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\varepsilon_j\int_{A_j} E_m(t_i, s)ds\right)^2$$

$$\hat{=}\ 2I_2^{(1)} + 2I_2^{(2)}. \tag{4.19}$$

By Lemma 4.2 and $n^{1/4}\tau_m \to 0$, we have that

$$\sqrt{n} I_2^{(1)} = O\left(n^{1/2 - 2\gamma}\right) + O\left(\sqrt{n}\tau_m^2\right) \to 0. \tag{4.20}$$

Let $\xi_n = \sum_{j=1}^{n}\hat{\varepsilon}_j = \sum_{j=1}^{n}\frac{1}{\sqrt{n}}\varepsilon_j\int_{A_j} E_m(t_i,\ s)ds$. Then $\{\xi_n,\ F_n,\ n \ge 1\}$ is a L_2-martingale. In fact, since

$$E\xi_n^2 = E\left(\sum_{j=1}^{n}\frac{1}{\sqrt{n}}\varepsilon_j\int_{A_j} E_m(t_i,\ s)ds\right)^2 \le \sum_{j=1}^{n} E\varepsilon_j^2\left(\int_{A_j} E_m(t_i,\ s)ds\right)^2$$

$$\le \sup_j E\varepsilon_j^2 \cdot \sup_j\left|\int_{A_j} E_m(t_i, s)ds\right| \cdot \left|\sum_{j=1}^{n}\int_{A_j} E_m(t_i, s)ds\right| \le C2^m n^{-1} < \infty.$$

It is easily seen that $\widehat{\varepsilon}_j = \frac{1}{\sqrt{n}} \varepsilon_j \int_{A_j} E_m(t_i, s) ds$ is a martingale difference sequence. Hence $\{\xi_n, F_n, n \geq 1\}$ is a L_2 - martingale. Thus, by Lemma 4.3, we have that

$$\begin{aligned}
\sqrt{n} E I_2^{(2)} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\int_{A_j} E_m(t_i, s) ds \right)^2 E\varepsilon_j^2 \\
&\leq C\sqrt{n} \cdot \sup_j E\varepsilon_j^2 \cdot \sup_i \sum_{j=1}^{n} \left(\int_{A_j} E_m(t_i, s) ds \right)^2 \\
&\leq C\sqrt{n} \cdot \sup_i \sup_j \left| \int_{A_j} E_m(t_i, s) ds \right| \cdot \left| \sum_{j=1}^{n} \int_{A_j} E_m(t_i, s) ds \right| \\
&\leq C 2^m n^{-1/2} = C\left(2^{2m} n^{-1}\right)^{1/2} \to 0.
\end{aligned}$$

Hence

$$\sqrt{n} I_2^{(2)} \xrightarrow{\quad p \quad} 0. \tag{4.21}$$

From (4.19)-(4.21), we obtain that

$$\sqrt{n} I_2 \xrightarrow{\quad p \quad} 0. \tag{4.22}$$

Let $A_n = \left(\widetilde{X}^T \widetilde{X}\right)^{-1/2} \widetilde{X}^T = \left(a_{nij}\right)_{d \times n}$. Since C_n is a symmetry matrix with $C_n^2 = C_n$, we obtain that $\sum_{k=1}^{n} a_{nik} a_{njk} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$, and $C_n = A_n^T A_n$. Thus

$$\begin{aligned}
I_3 &= 2/n(g - \hat{g}_0)^T \varepsilon - 2/n(g - \hat{g}_0)^T C_n \varepsilon \\
&= \frac{2}{n} \sum_{i=1}^{n} (g(t_i) - \hat{g}_0(t_i)) \varepsilon_i - \frac{2}{n} \sum_{j=1}^{d} \left(\sum_{i=1}^{n} a_{nji} \varepsilon_i \right) \left(\sum_{i=1}^{n} a_{nji} (g(t_i) - \hat{g}_0(t_i)) \right) \\
&= 2I_3^{(1)} - 2I_3^{(2)}.
\end{aligned} \tag{4.23}$$

$$I_3^{(1)} = \frac{1}{n}\sum_{i=1}^{n}\left(g(t_i) - \sum_{j=1}^{n} g(t_j)\int_{A_j} E_m(t_i, s)ds\right)\varepsilon_i - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_i\varepsilon_j\int_{A_j} E_m(t_i, s)ds$$

$$= I_3^{(11)} + I_3^{(12)}. \tag{4.24}$$

Let $\Gamma_n = \sum_{i=1}^{n}\frac{1}{n}\left(g(t_i) - \sum_{j=1}^{n} g(t_j)\int_{A_j} E_m(t_i, s)ds\right)\varepsilon_i$. Then $\{\Gamma_n, F_n, n \geq 1\}$ is a L_2-martingale. In fact, using C_r-inequality and Lemma 4.2, we obtain

$$E\Gamma_n^2 \leq \frac{1}{n^2}\cdot n \cdot \sum_{i=1}^{n}\left(g(t_i) - \sum_{j=1}^{n} g(t_j)\int_{A_j} E_m(t_i, s)ds\right)E\varepsilon_i^2$$

$$\leq \sup_i E\varepsilon_i^2 \cdot \sup_i\left|g(t_i) - \sum_{j=1}^{n} g(t_j)\int_{A_j} E_m(t_i, s)ds\right|^2 < \infty.$$

It is easy to see that $\frac{1}{n}\left(g(t_i) - \sum_{j=1}^{n} g(t_j)\int_{A_j} E_m(t_i, s)ds\right)\varepsilon_i$ is a martingale difference sequence, so $\{\Gamma_n, F_n, n \geq 1\}$ is a L_2-martingale. Thus, by Lemma 4.2, we have

$$nE\left(I_3^{(11)}\right)^2 \leq \sup_i E\varepsilon_i^2 \cdot \sup_i\left|g(t_i) - \sum_{j=1}^{n} g(t_j)\int_{A_j} E_m(t_i, s)ds\right|^2 \to 0.$$

That is,

$$\sqrt{n}I_3^{(11)} \xrightarrow{\ p\ } 0. \tag{4.25}$$

It is easily seen that $\left\{\sum_{i=1}^{n}\varepsilon_i \Big/ \sqrt{n},\ F_n,\ n \geq 1\right\}$ is a L_2-martingale, so we have that

$$\sqrt{n}EI_3^{(12)} \leq \sqrt{n}\sup_{i,j}\left|\int_{A_j} E_m(t_i, s)ds\right| \cdot \sum_{i\neq j} E\left(\frac{\varepsilon_i}{\sqrt{n}}\frac{\varepsilon_j}{\sqrt{n}}\right)$$

$$+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}E\varepsilon_i^2\int_{A_j}E_m(t_i, s)ds$$

$$\leq C\left(2^{2m}n^{-1}\right)^{1/2}\cdot 0+\sqrt{n}\cdot\sup_i E\varepsilon_i^2\cdot\sup_i\left|\int_{A_j}E_m(t_i, s)ds\right|$$

$$\leq C2^m n^{-1/2} = C\left(2^{2m}n^{-1}\right)^{1/2}\to 0.$$

That is,

$$\sqrt{n}I_3^{(12)}\xrightarrow{p}0. \tag{4.26}$$

By (4.24)-(4.26), we have that

$$\sqrt{n}I_3^{(1)}\xrightarrow{p}0. \tag{4.27}$$

$$I_3^{(2)} = \frac{1}{n}\sum_{j=1}^{d}\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)\left(\sum_{i=1}^{n}a_{nji}\left(g(t_i)-\sum_{k=1}^{n}g(t_k)\int_{A_k}E_m(t_i, s)ds\right)\right)$$

$$-\frac{1}{n}\sum_{j=1}^{d}\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)\left(\sum_{i=1}^{n}a_{nji}\left(\sum_{k=1}^{n}\varepsilon_k\int_{A_k}E_m(t_i, s)ds\right)\right) = I_3^{(21)}-I_3^{(22)}. \tag{4.28}$$

Using C_r -inequality and Lemma 4.2, we obtain

$$n\left(I_3^{(21)}\right)^2 \leq \frac{C}{n}\sup_{1\leq j\leq d}\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)^2\left(\sum_{i=1}^{n}a_{nji}\left(g(t_i)-\sum_{k=1}^{n}g(t_k)\int_{A_k}E_m(t_i, s)ds\right)\right)^2$$

$$\leq\frac{C}{n}\cdot n\cdot\sup_{i, j}\left|g(t_i)-\sum_{k=1}^{n}g(t_k)\int_{A_k}E_m(t_i, s)ds\right|^2\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)^2\sum_{i=1}^{n}a_{nji}^2$$

$$\leq C\left(n^{\frac{1}{2}-2\gamma}+n^{\frac{1}{2}}\tau_m^2\right)\sup_j\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)^2. \tag{4.29}$$

It is easy to show that

$$\sqrt{n}I_1^{(2)} = \frac{1}{\sqrt{n}}\varepsilon^T C_n\varepsilon = \frac{1}{\sqrt{n}}(A_n\varepsilon)^T(A_n\varepsilon) = \frac{1}{\sqrt{n}}\sum_{j=1}^{d}\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)^2 \xrightarrow{\ p\ } 0. \quad (4.30)$$

Therefore, by (4.29), (4.30) and $n^{1/4}\tau_m \to 0$, we obtain that $n\left(I_3^{(21)}\right)^2 \xrightarrow{\ p\ } 0.$ Hence,

$$\sqrt{n}I_3^{(21)} \xrightarrow{\ p\ } 0. \quad (4.31)$$

By C_r -inequality, we obtain

$$n\left(I_3^{(22)}\right)^2 \le \frac{C}{\sqrt{n}}\sup_{1\le j\le d}\left(\sum_{i=1}^{n}a_{nji}\left(\sum_{k=1}^{n}\varepsilon_k\int_{A_k}E_m(t_i,\, s)ds\right)\right)^2 \cdot \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)^2$$

$$\le C\sqrt{n}\sup_{i,\,j}\left(\sum_{k=1}^{n}\varepsilon_k\int_{A_k}E_m(t_i,\, s)ds\right)^2\sum_{i=1}^{n}a_{nji}^2 \cdot \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)^2$$

$$\le Cn\sup_{i}\left(\sum_{k=1}^{n}n^{-1/4}\varepsilon_k\int_{A_k}E_m(t_i,\, s)ds\right)^2 \cdot \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\sum_{i=1}^{n}a_{nji}\varepsilon_i\right)^2. \quad (4.32)$$

It is easy to prove that $\Gamma_n = \sum_{k=1}^{n}n^{-1/4}\varepsilon_k\int_{A_k}E_m(t_i,\, s)ds$ is a L_2-martingale, so we obtain

$$nE\left(\sum_{k=1}^{n}n^{-1/4}\varepsilon_k\int_{A_k}E_m(t_i,\, s)ds\right)^2 = \sqrt{n}\sum_{k=1}^{n}E\varepsilon_k^2\left(\int_{A_k}E_m(t_i,\, s)ds\right)^2$$

$$\le C\sqrt{n}\cdot\sup_{k}E\varepsilon_k^2\cdot\sup_{k}\left|\int_{A_k}E_m(t_i,\, s)ds\right|\cdot\left|\sum_{k=1}^{n}\int_{A_k}E_m(t_i,\, s)ds\right|$$

$$\le C2^m n^{-1/2} = C\left(2^{2m}n^{-1}\right)^{1/2} \to 0. \quad (4.33)$$

By (4.30)-(4.33), we obtain that

$$\sqrt{n}I_3^{(22)} \xrightarrow{\ p\ } 0 \ (n \to \infty). \quad (4.34)$$

By (4.28), (4.31) and (4.34), we obtain

$$\sqrt{n} I_3^{(2)} \xrightarrow{\quad p \quad} (n \to \infty). \tag{4.35}$$

By (4.23), (4.27) and (4.35), we obtain

$$\sqrt{n} I_3 \xrightarrow{\quad p \quad} 0 \ (n \to \infty). \tag{4.36}$$

Thus, by (4.3), (4.18), (4.22) and (4.36), the proof of Theorem 3.1 is completed. $\square$

Proof of Theorem 3.2. The result follows immediately from Lemma 4.7 and Theorem 3.1. $\square$

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